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Two-spin entanglement distribution near factorized states

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Abstract
We study the two-spin entanglement distribution along the infinite $S = 1/2$ chain described by the XY model in a transverse field; closed analytical expressions are derived for the one-tangle and the concurrences C_r, with r being the distance between the two possibly entangled spins, for the values of the Hamiltonian parameters close to those corresponding to factorized ground states. The total amount of the entanglement, the fraction of such entanglement which is stored in pairwise entanglement and the way such fraction distributes along the chain are discussed, with attention focused on the dependence on the anisotropy of the exchange interaction. Near factorization a characteristic length scale naturally emerges in the system, which is specifically related with the entanglement properties and diverges at the critical point of the fully isotropic model. In general, we find that anisotropy rules a complex behavior of the entanglement properties, which results in the fact that more isotropic models, despite being characterized by a larger amount of the total entanglement, present a smaller fraction of the pairwise entanglement: the latter, in turn, is more evenly distributed along the chain, to the extent that, in the fully isotropic model at the critical field, the concurrences do not depend on r.

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1. Introduction

The analysis of the entanglement properties has recently furnished new insights into several peculiar features of many-body systems, such as the occurrence of quantum phase transitions or that of non-trivial factorized ground states [1–11]. Different types of the entanglement can be defined in many-body systems, but computable measures are available just for a few of them. In this sense, a privileged role is played by the bipartite entanglement of formation [12], and by the related quantities, one-tangle and concurrence, which represent the entanglement of formation between one qubit and the rest of the system, and that between two selected qubits of the system, respectively. In particular, the definition of the concurrence holds not only for pure states, as in the case of the one-tangle, but also for mixed ones [13, 14].

When magnetic systems are considered, the qubit is naturally represented by a spin with $S = 1/2$: for interacting magnetic models described by Hamiltonians with certain symmetry properties, both the one-tangle and the concurrence are expressed in terms of standard magnetic observables, such as the magnetizations and the correlation functions, making feasible a quantitative analysis of the entanglement dependence on the Hamiltonian parameters. If analytical expressions are available, a general discussion of such a dependence is at hand, which is the reason why a renewed interest is being devoted to exactly solvable models.

In this paper, we focus on the $T = 0$ behavior of the best known one-dimensional XY model in a transverse field, in the vicinity of factorized ground states [10, 15]. We derive closed analytical formulae for the magnetization and the correlators, as functions of the anisotropy, the field and the distance between the two selected spins, which allow us to study the long-distance behavior of the concurrence as the Hamiltonian parameters are varied. By using these expressions for the correlation functions, we explicitly proof the divergence of the range of the concurrence in the anisotropic model (whose preliminary result was presented in [16]) and we extend it in the whole parameter space, studying the fully isotropic case as well as the slightly anisotropic region close to the spin saturation. Our analysis, besides analytically confirming the divergence of the range of the concurrence, shows that to such divergence corresponds the appearance of a characteristic length scale in the system that we have named *two-spin entanglement length*. This length scale depends on the value of the anisotropy and keeps finite as far as the model belongs to the Ising universality class, while diverging when factorization gets to coincide with saturation, i.e. for the isotropic XX model, which belongs to the Kosterlitz–Thouless universality class. Correspondingly, the way the two-spin entanglement distributes along the chain is found to strongly depend on the symmetry of the model: in the XX model a good amount of the entanglement can be stored even between two spins which are far apart from each other, while in the Ising model the pairwise entanglement of the ground state is shared only between nearest and next-nearest neighboring spins.

The structure of the paper is as follows: in section 2 we introduce the model and the entanglement properties we aim at studying; in section 3 we study the long-distance pairwise entanglement both in the anisotropic and in the isotropic case; In section 4 we define and analyze the two-spin entanglement length, while in section 5 we use our results to understand the interplay between the pairwise entanglement and the multipartite entanglement. Finally, in section 6, we draw the conclusions.

2. Model

The XY model in a transverse field is described by the Hamiltonian

$$H = J \sum_i \left[(1+\gamma) S_i^x S_{i+1}^x + (1-\gamma) S_i^y S_{i+1}^y - h S_i^z\right], \tag{1}$$

where i runs over the sites of an infinite chain, S_i^η ($\eta = x, y, z$) are the $S = 1/2$ quantum spin operators, $\gamma \in [0, 1]$ is the anisotropy and $h = g\mu_B H/J$ is the reduced magnetic field, where $J > 0$ is the strength of the exchange interaction.

For $0 < \gamma \leqslant 1$, the model belongs to the Ising universality class and at $T = 0$ the critical field $h_c = 1$ separates a disordered phase ($h > h_c$), from a spontaneously broken-symmetry phase, where the staggered order parameter is finite $\left(\langle S_i^x\rangle \neq 0\right)$. In the isotropic case, $\gamma = 0$, the model has an additional rotational symmetry on the xy plane, and the critical field coincides with the saturation field, above which all the spins incoherently align parallel to the field. For $\gamma = 0$ and $h \geqslant 1$, the system is in a fully polarized phase $\left(\langle S_i^z\rangle = \frac{1}{2}\right)$, while for $h < h_c$, the system is in a gapless phase with $\langle S_i^z\rangle < \frac{1}{2}, \langle S_i^x\rangle = 0$ and power-law decaying correlation functions in the xy plane. No spontaneous symmetry breaking is present in the isotropic case, as testified by $\langle S_i^x\rangle$ being null for whatever value of the applied field.

Let us now consider the $h - \gamma$ parameter space of the Hamiltonian (1): the ground state of the model is exactly factorized [15]:

$$|\mathrm{GS}\rangle = \prod_i |\phi_i\rangle, \tag{2}$$

on the circle $h^2+\gamma^2 = 1$, as well as along the line $\{h \geqslant 1, \gamma = 0\}$: such a ground state has a Nèel structure given by $|\phi_i\rangle = (-1)^i \cos\theta_\gamma|\uparrow_i\rangle + \sin\theta_\gamma|\downarrow_i\rangle$, with $\cos\theta_\gamma = \sqrt{(1-\gamma)/(1+\gamma)} \equiv \alpha$, which reduces to the trivial ferromagnetic ground state for $\gamma = 0$ and $h \geqslant 1$. In what follows, we will refer to the circle $h^2 + \gamma^2 = 1$ as the *factorized circle* and to the line $\{\gamma = 0, h \geqslant 1\}$ as the *factorized line*.

For the model, equation (1), the concurrence C_r between two spins sitting on sites i and j, with $|i - j| = r$, reads [17]

$$C_r = 2\max\{0, C'_r, C''_r\}, \tag{3}$$

$$C'_r = \left|g_r^{xx} + g_r^{yy}\right| - \sqrt{\left(\tfrac{1}{4} + g_r^{zz}\right)^2 - M_z^2}, \tag{4}$$

$$C''_r = \left|g_r^{xx} - g_r^{yy}\right| + g_r^{zz} - \tfrac{1}{4} \tag{5}$$

while the one-tangle is $\tau_1 = 1 - 4\left(M_x^2 + M_z^2\right)$, with the correlators $g_{ij}^{\eta\eta} = \langle S_i^\eta S_j^\eta\rangle$ and the magnetizations $M_\eta = \langle S_i^\eta\rangle$. The terms C'_r and C''_r, equations (4) and (5), are related to the probabilities for the two considered spins to be either in antiparallel or in parallel Bell states, respectively [18]. The total amount of the bipartite entanglement may be estimated by the so-called *two-tangle*, namely the sum $\tau_2 = 2\sum_r C_r^2$, which is related with the one-tangle via the monogamy inequality $\tau_2 \leqslant \tau_1$ [19, 20]. The difference $\tau_1 - \tau_2$ is the so-called *residual tangle*, while the ratio τ_2/τ_1 is usually referred to as the *entanglement ratio*.

Equations (3)–(5) have been originally derived [17] exploiting the symmetries of the Hamiltonian equation (1), for this reason in the presence of spontaneous symmetry breaking, i.e. for $\gamma > 0$ and $h < 1$, one has to be careful. This problem has been studied in [21, 22]: equations (3)–(5) generally hold when $C''_r > C'_r$, i.e. for $h^2 + \gamma^2 > 1$ [16], while in the antiparallel region $h^2 + \gamma^2 < 1$ where $C'_r > C''_r$ they represent a lower bond for the pairwise entanglement. Moreover, from equation (7) of [22], one can see that in the asymptotic limit $r \to \infty$ they also stay valid for $C'_r > C''_r$, which makes the analysis of the long-distance concurrence, reported below, valid both inside and outside the factorized circle. As for the XX model, no spontaneous symmetry breaking occurs for whatever value of the field.

3. Long-distance concurrence

3.1. Anisotropic case

Let us first consider the behavior of the model for $\gamma > 0$ and $h > h_\mathrm{f}$, in the vicinity of the factorized circle: we will keep fixed and finite the value of γ, and vary the field, meaning that we will move along horizontal lines in the $h - \gamma$ plane.

The $T = 0$ correlation functions entering the expressions of C_r for the XY model in a transverse field are usually evaluated numerically, by computing the corresponding Toeplitz determinants, and cannot be written in closed form for generic r, except in the case of factorized ground states, where they do not depend on r. Since we are interested in the behavior of the concurrence as the factorized circle is approached, we fix the value of γ and derive C_r as a series expansion in the difference $h - h_{\mathrm{f}}$, with the factorizing field $h_{\mathrm{f}} = \sqrt{1-\gamma^2}$. The entries of Toeplitz determinant are basically given by the well-known G function [23]

$$G(r, h, \gamma) = \frac{1}{\pi}\int_0^{\pi} \mathrm{d}\phi \frac{(h - \cos\phi)\cos(r\phi) + \gamma \sin\phi \sin(r\phi)}{\lambda(h, \gamma; \phi)}, \tag{6}$$

with $\lambda(h, \gamma; \phi) = \sqrt{(h - \cos\phi)^2 + \gamma^2 \sin^2\phi}$, and we have to expand it in the difference $h - h_{\mathrm{f}}$, thus obtaining

$$\begin{aligned} G(r, h, \gamma) = {} & \frac{1}{\pi}\int_0^{\pi} \mathrm{d}\phi \frac{(\sqrt{1-\gamma^2} - \cos\phi)\cos(r\phi) + \gamma \sin\phi \sin(r\phi)}{\lambda_{\mathrm{f}}(\phi)} \\ & + \frac{1}{\pi}\int_0^{\pi} \mathrm{d}\phi \left[\frac{\cos(r\phi)}{\lambda_{\mathrm{f}}(\phi)} \left(1 - \frac{(\sqrt{1-\gamma^2} - \cos\phi)^2}{\lambda_{\mathrm{f}}^2(\phi)} \right) \right. \\ & \left. - \frac{(\sqrt{1-\gamma^2} - \cos\phi)\gamma \sin\phi \sin(r\phi)}{\lambda_{\mathrm{f}}^3(\phi)} \right] (h - h_f) + \cdots, \end{aligned} \tag{7}$$

with $\lambda_{\mathrm{f}}(\phi) = \lambda(h_{\mathrm{f}}, \gamma; \phi)$. The trigonometric functions of the angle $r\phi$ can be represented in terms of powers of these functions, then, integrating by part and resumming all the r terms, one gets [24]

$$\begin{aligned} G(r, h, \gamma) &= \frac{1}{2\gamma}\alpha^r (h - h_{\mathrm{f}}) + O(h - h_{\mathrm{f}})^2, \\ G(0, h, \gamma) &= \alpha + \frac{1}{2\gamma}(h - h_{\mathrm{f}}) + O(h - h_{\mathrm{f}})^2, \\ G(-r, h, \gamma) &= -\frac{2\gamma}{1+\gamma}\alpha^{-r-1} + \frac{1 + 2r\gamma - (2r^2+1)\gamma^2}{2\gamma(\gamma+1)^2}\alpha^{r-2}(h - h_{\mathrm{f}}) + O(h - h_{\mathrm{f}})^2, \end{aligned} \tag{8}$$

where the first and third equations hold for $r \neq 0$. From equations (8), the expansion for the correlators is found in closed form:

$$g_r^{xx} = \frac{(-1)^r}{4}\left[\frac{2\gamma}{1+\gamma} + \frac{\alpha^{2r+1} - 2\alpha}{2\gamma}(h - h_{\mathrm{f}})\right] + O(h - h_{\mathrm{f}})^2, \tag{9}$$

$$g_r^{yy} = -\frac{(-1)^r}{4}\frac{\alpha^{2r-1}}{2\gamma}(h - h_{\mathrm{f}}) + O(h - h_{\mathrm{f}})^2, \tag{10}$$

$$g_r^{zz} = \frac{1}{4}\left[\alpha^2 + \left(\frac{\alpha}{\gamma} + \frac{\alpha^{2r-1}}{\gamma+1}\right)(h - h_{\mathrm{f}})\right] + O(h - h_{\mathrm{f}})^2, \tag{11}$$

and the magnetization along the field direction reads

$$M_z = \frac{\alpha}{2} + \frac{1}{4\gamma}(h - h_{\mathrm{f}}) + O(h - h_{\mathrm{f}})^2. \tag{12}$$

In the most anisotropic $\gamma = 1$ case, it is $\alpha = 0$ and the only finite correlator up to the first order in $(h - h_{\mathrm{f}})$ is g_r^{xx}, whose modulus gets the maximum value (i.e. $|g_r^{xx}| = 1/4$) independently of h and r; the first correction is of order $(h - h_{\mathrm{f}})^2$, being $\left|g_r^{xx}\right| = 1/4 - (h - h_{\mathrm{f}})^2/16$. Note that equations (9)–(12) do not hold for $\gamma = 0$, where in fact they display unphysical singularities.

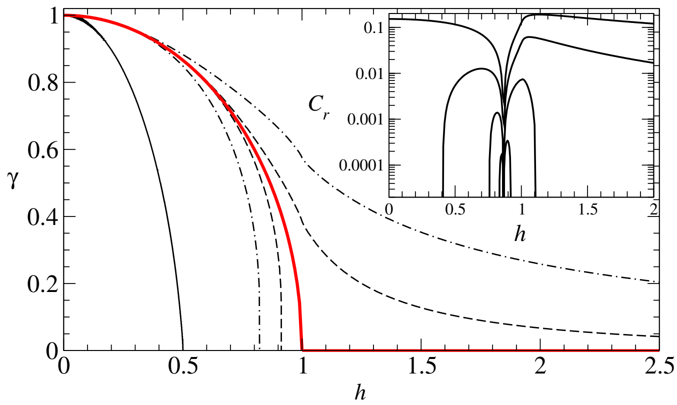

Figure 1. Entanglement phase diagram: the thick curve is the line where the ground state is exactly factorized. The regions between curves with the same drawing represent the areas of the $h - \gamma$ plane where $C_r \neq 0$ for $r = 3$ (dot-dashed) and $r = 4$ (dashed). $C_2 = 0$ only below the solid line and at $h = h_{\mathrm{f}}$ and C_1 vanishes only at $h = h_{\mathrm{f}}$. The inset shows C_r versus h for different $r = 1, \ldots, 4$ (from top to bottom) at $\gamma = 0.5$.

(This figure is in colour only in the electronic version)

Similar expressions are found for $h < h_{\mathrm{f}}$ and, by defining the *distance* $\varepsilon \equiv |h - h_{\mathrm{f}}|$, we obtain the first-order term of the expansion in ε for C_r, which reads

$$C_r = \frac{\alpha^{2r-1}}{2\gamma}\varepsilon + O(\varepsilon^2), \tag{13}$$

and noticeably holds for whatever r.

Let us now focus on the range R of the concurrence [16, 18], which is the distance between the two farthest entangled spins along the chain, i.e.,

$$R\colon \quad C_r > 0, \qquad \forall r \leqslant R \wedge C_r = 0, \qquad \forall\, r > R. \tag{14}$$

Since C_r is finite for all r at the first order in ε, R diverges for $\varepsilon \to 0$. This statement is compatible with the behavior depicted by the exact numerical data shown in figure 1; from the same data, we also see that for whatever $r > 1$, there exists a distance $\varepsilon_0(r, \gamma)$ such that $C_r > 0$ for $0 < \varepsilon < \varepsilon_0(r, \gamma)$. On the other hand, since equation (13) cannot describe the vanishing of C_r at $\varepsilon_0(r, \gamma)$, in order to further analyze the behavior of R, one has to evaluate the concurrence up to the second order in ε. As we are interested in the behavior of the long-distance concurrence, we are allowed to use the large-r asymptotic expressions of the correlators [23], thus finding

$$\frac{C_r}{2} = \frac{\alpha^{2r-1}}{4\gamma}\varepsilon - [A^2 - \delta A^2(r)]\varepsilon^2 + O(\varepsilon^3), \tag{15}$$

where $A^2 = \alpha^2(\gamma + 3)/32\gamma^3$ and $\delta A^2(r) \sim O(r^{-2})$. It is important to note that, in contrast to equation (13), the above expression only holds for large r. The behavior predicted by equation (15), though approximated, is consistent with that shown by the exact numerical data (see figure 2). In particular, beyond the trivial zero in $\varepsilon = 0$, equation (15) has another zero which approximates $\varepsilon_0(r, \gamma)$:

$$\varepsilon_0(r, \gamma) \simeq \frac{\alpha^{2r-1}}{4\gamma A^2}, \tag{16}$$

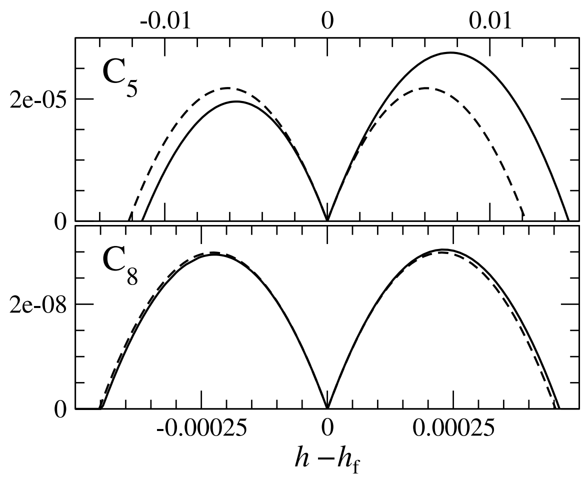

Figure 2. C_r versus $h - h_f$, for $\gamma = 0.5$, $r = 5$ (top panel) and $r = 8$ (bottom panel): comparison between the exact (full line) and approximated (equation (15) with $\delta A^2(r) = 0$) value (dashed line).

where we neglected the $\delta A^2(r)$ term, which vanishes for $r \to \infty$. Note that the symmetry of equations (15) and (16) with respect to the sign of the difference $h - h_f$ arises from the second-order expansion in $h - h_f$ itself that becomes more and more accurate for larger r (see figure 2).

For a given (large) r, $\varepsilon_0(r, \gamma)$ is the distance from h_f at which C_r gets finite while approaching the factorizing field. We can rephrase this statement by saying that, for fixed $h \neq h_f$, the farthest entangled spins are those whose distance r fulfils equation (16), with $\varepsilon_0 = |h - h_f|$. Therefore, if we consider r as a continuous variable, equation (16) can be inverted and we obtain, for the range of the concurrence defined in equation (14), the following expression:

$$R \simeq \frac{1}{\ln \alpha^2} \ln \varepsilon + \frac{\ln(4\alpha\gamma A^2)}{\ln \alpha^2} \qquad \text{for} \quad \varepsilon \to 0. \tag{17}$$

From the above expression we see that, for a fixed value of ε, a larger anisotropy implies a smaller value of R. On the other hand, when the anisotropy increases, one should pay particular attention to the overall consistency of the reasoning, as the existence itself of ε_0 is not generally due for small r, given that equation (17) holds only for large r. Specifically, for $h < h_f$ C_1 is always finite, and for $h > h_f$ both C_1 and C_2 keep finite no matter the value of the field, as seen in figure 1. In particular, for $\gamma = 1$ the above scheme breaks down: for any finite value of the field, the only non-zero concurrences are those between nearest and next-nearest neighbors, as from the exact results by Pfeuty [25], which give

$$C_1 = \frac{h^2}{8} + \frac{3h^4}{128} + O(h^6); \qquad C_2 = \frac{h^4}{128} + O(h^6).$$

3.2. Isotropic case

Factorization and quantum criticality are two distinct phenomena, occurring usually for different values of the external magnetic field, being typically $0 < h_f < h_c$. When factorization occurs, peculiar features of the two-spin entanglement distribution are observed at h_f where, in turn, standard magnetic observables behave quite trivially. On the other hand, it is just the peculiar behavior of these latter properties that signals the occurrence of a quantum phase transitions at h_c, where two-spin entanglement distribution has no distinctive features.

When factorization and quantum criticality get to coincide at $h_f = h_c$, standard two-point correlation functions and pairwise entanglement together signal the occurrence of a phenomenon which corresponds both to a factorization (though of a particular type, i.e. saturation) and a quantum phase transition (though of topological type rather than second order).

The specificity of the above-depicted situation translates into a peculiar distribution of the two-spin entanglement along the chain, which gives the isotropic XX model a special role in the overall analysis, as shown below.

Let us consider equation (1) with $\gamma = 0$, in the non-trivial quasi-ordered phase, $h < h_f = 1$: using a procedure similar to that depicted above, we obtain the following closed forms for the expansions of the correlators:

$$g_r^{xx(yy)} = (-1)^r \left[\frac{\varepsilon^{1/2}}{\pi\sqrt{2}} - \frac{(4r^2-1)}{12\pi\sqrt{2}}\varepsilon^{3/2} + \frac{2r(r^2-1)}{9\pi^2}\varepsilon^2 \right] + O(\varepsilon^{5/2}), \tag{18}$$

$$g_r^{zz} = \frac{1}{4} - \frac{\sqrt{2}}{\pi}\varepsilon^{1/2} - \frac{\varepsilon^{3/2}}{6\pi\sqrt{2}} + \frac{4r^2}{3\pi^2}\varepsilon^2 + O(\varepsilon^{5/2}), \tag{19}$$

where $g_r^{xx} = g_r^{yy}$ due to the symmetry in the xy plane, and the magnetization along the field direction reads

$$M_z = \frac{1}{2} - \frac{1}{\pi}\cos^{-1} h. \tag{20}$$

The expansion of the concurrence consequently reads

$$C_r = \frac{2\sqrt{2}}{\pi}\varepsilon^{1/2} - \frac{4r}{\pi\sqrt{3}}\varepsilon + \frac{8r\sqrt{3} - (4r^2-1)\pi}{3\pi^2\sqrt{2}}\varepsilon^{3/2} + 2r\frac{30\sqrt{3} + 20(r^2-1)\pi + (4r^2-5)\sqrt{3}\pi^2)}{45\pi^3}\varepsilon^2 + O(\varepsilon^{5/2}). \tag{21}$$

From the above expressions, we see that the change of the universality class at $\gamma = 0$ has drastic effect on the mechanism of rearrangement of two-spin entanglement along the chain. In particular, the fact that the correlation functions have an algebraic dependence on r, rather than the exponential one found in the anisotropic case, reflects in the independence of r of the first term of equation (21). Thus, whatever the selection of the two spins in the chain they share the same amount of the entanglement. Such an invariance is quite a surprising feature and it suggests, according to the analysis proposed in [26], the more relevant role of the multipartite entanglement in the isotropic case.

Moreover, the comparison between equations (21) and (15), as well as that between the inset of figure 1 and figure 3, evidences an overall increase of *all* the concurrences $\{C_r\}$ in the most isotropic case, consistently with what is observed in the XXZ model [5].

Let us now study R as saturation is approached from below, i.e. for $h \to h_f^-$: its divergence is favored by the increase of the symmetry, as testified by the singularity of the prefactor $1/\ln\alpha^2$ in equation (17) as $\gamma \to 0$, that signals a qualitative change in the behavior of R. In contrast to the anisotropic case, for $\gamma = 0$ we already got C_r to order ε^2 (equation (21)) and we do not have to resume the $1/r$ asymptotic expansions of the correlators to evaluate the range of the concurrence. The farthest entangled spins are those whose distance r fulfils equation (21) $= 0$; the latter is an equation of the third order in r that for sufficiently small ε has three real solutions, the smallest positive one is just R. As in previous large-r studies based

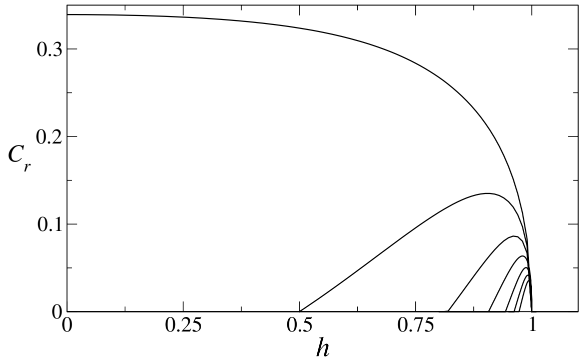

Figure 3. Concurrence C_r versus h for $\gamma = 0$ (XX model) and $r = 1, \ldots, 7$ (from the highest to the lowest curve).

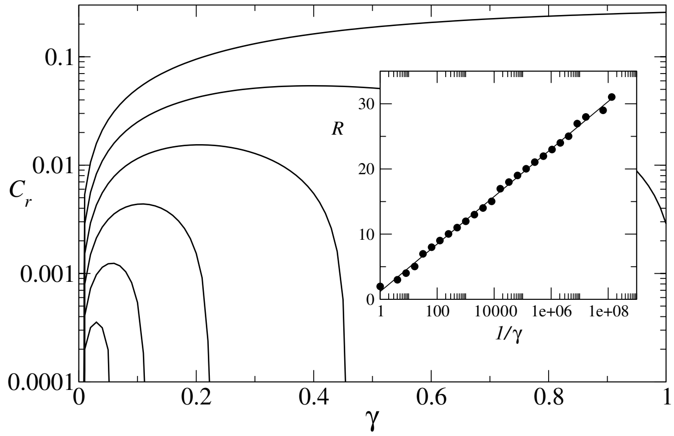

Figure 4. C_r versus γ for $h = 1.2$ and $r = 1, \ldots, 6$ (from the highest to the lowest curve). The inset shows the logarithmic divergence of R for $\gamma \to 0$.

on the XXZ model [16, 18], we find that R diverges more rapidly than in the anisotropic case (equation (17)), namely

$$R \propto \varepsilon^{-1/2}. \tag{22}$$

For $h > 1$, being the ground state factorized, all the entanglement measures vanish; however one may fix a value of h larger than unity and study the behavior of the concurrences as $\gamma \to 0$. We therefore computed the correlators and the concurrences in this parameter region as a function of the anisotropy. In figure 4, $\{C_r\}_{r=1,\ldots,6}$ are plotted versus γ at fixed field. Again we see that, while approaching the factorized ground state, i.e. for $\gamma \to 0$, all $\{C_r\}$ get finite and the range of the concurrence diverges with a logarithmic trend: $R \propto 1/\ln\gamma$. Our results show that this behavior is general for any $h > 1$ and the divergence becomes more and more pronounced for $h \to 1$. Eventually, as shown by figure 5, at the saturation field $h_\mathrm{f} = h_\mathrm{c} = 1$ the divergence of the concurrence range modifies its dependence on the anisotropy, being $R \propto 1/\gamma$. Thus, the change of the character of the divergence of R from logarithmic to power law observed both for $\gamma = 0$ and $h \to h_\mathrm{f}^-$ and for $h = h_\mathrm{f}$ and $\gamma \to 0$ suggests the critical point of the isotropic model to represent a peculiar point in this context, as also proposed in [27], though in a slightly different sense.

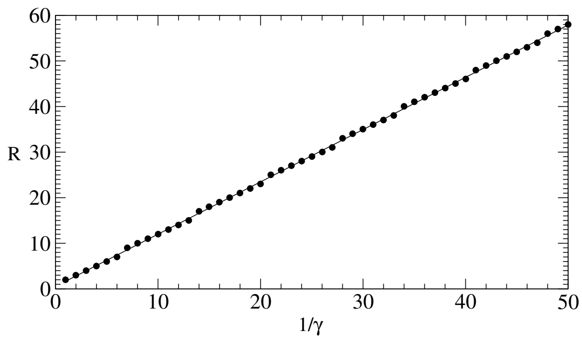

Figure 5. R versus $1/\gamma$ for $h = h_c = 1$.

4. Two-spin entanglement length ξ_{2SE}

The most noticeable feature of the first-order expansion in ε of C_r, equation (13), is the purely exponential dependence on r, $C_r \sim \alpha^{2r}$, which indicates that a characteristic length emerges in the system near factorized ground states. This length scale is

$$\xi_{2SE} \equiv -\frac{r}{\ln(2\alpha\gamma C_r)} = \frac{1}{|\ln \alpha^2|}, \tag{23}$$

and we have named it *two-spin entanglement length* as it specifically characterizes the distribution of the entanglement between different spin pairs along the chain. In fact, by looking at the expansions of the correlators, equations (9)–(11), we note that g_r^{yy} has the same purely exponential behavior of C_r, which means that ξ_{2SE} coincides with the standard correlation length along the less favored direction which, in turn, does not enter the characterization of the magnetic behavior. In fact, the relation between g_r^{yy} and C_r is not accidental: as a matter of fact the change of sign of g_r^{yy}, namely $g_r^{yy} < 0$ for $h < h_f$, $g_r^{yy} > 0$ for $h > h_f$ and $g_r^{yy} = 0$ at h_f, rules the swap between parallel and antiparallel entanglement, i.e. between C_r' and C_r'' (see equations (4) and (5)). So that surprisingly g_r^{yy}, which is the less significant correlator as far as the standard magnetic properties are concerned, plays a relevant role in determining spin pair entanglement properties close to the factorization.

The above definition, equation (23), makes sense only if equation (13) holds, i.e. for very small ε, where the concurrence is finite for any spin pair along the chain. In other terms, when R diverges C_r is found to decay exponentially with a characteristic length ξ_{2SE} which gets larger and larger as the anisotropy decreases, finally diverging as the isotropic critical point ($h = 1$, $\gamma = 0$) is approached along a vertical line, being

$$\xi_{2SE} \sim \frac{1}{2\gamma} \quad \text{for} \quad \gamma \to 0. \tag{24}$$

In order to further investigate the behavior of the isotropic model, let us extend our analysis to the vicinity of the factorized line: in the previous section we observed analogies between the behavior of the two-spin entanglement close to the factorized circle (i.e. $h < 1$) and close to the factorized line (i.e. $h > 1$). In particular, we found that also for $h > 1$ and $\gamma \to 0$ all the concurrences $\{C_r\}$ become finite, so that one can ask whether it is possible to extend the definition of ξ_{2SE} to the region $h > 1$ and $\gamma \ll 1$. We computed C_r versus r for a fixed value of $\gamma \ll 1$ and also in this case we found an exponential dependence on r, as shown by the inset of figure 6. Thus, extending definition (23) to the region $h > 1$, the two-spin entanglement length can be evaluated by the slope of the lines in a lin-log plot of C_r versus r. As the point

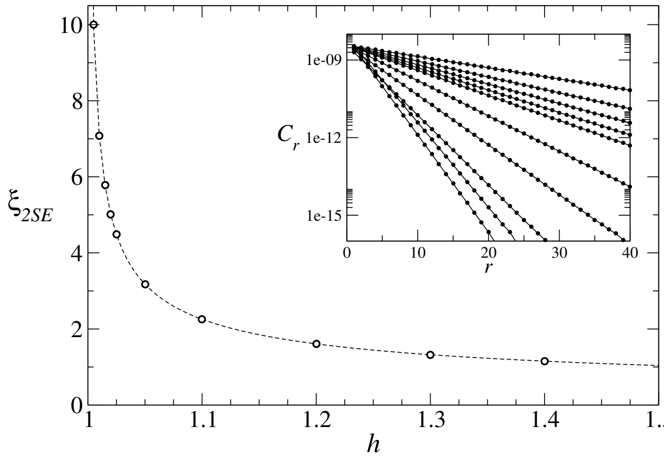

Figure 6. ξ_{2SE} versus h at $\gamma = 7.5 \times 10^{-9}$. The dashed line is the best fit $f(h) = 0.055 + 0.69/(h-1)^{0.50}$. In the inset, the lin-log plot of C_r versus r, the slopes of the lines correspond to the values of ξ_{2SE} in the main panel.

$\gamma = 0$ and $h = 1$ is approached from $h > 1$, our numerical results show that

$$\xi_{2SE} \propto \frac{1}{(h-1)^{\nu}} \quad \text{for} \quad h \to 1^{+}, \tag{25}$$

with $\nu = 0.50$. We note that, by using the identity $\gamma = \sqrt{1 - h_f^2}$, one may recast equation (24) in the form $\xi_{2SE} \propto 1/\left[2\left(1-h_f^2\right)^{1/2}\right] \sim 1/[2\sqrt{2}(1-h_f)^{1/2}]$ which reproduces the behavior of the above equation (25): this tells us that the way the two-spin entanglement length diverges while approaching the ($\gamma = 0$, $h = 1$) critical point does not depend on whether one moves h or γ.

The divergence of the two-spin entanglement length for $\gamma \to 0$ and $h \to 1$ means that in the neighborhood of this point, not only all the concurrences $\{C_r\}$ are finite for any r, but also that the pairwise entanglement does not depend on the distance r between spins, as the expression for the concurrences equation (21) anticipated. Elsewhere the concurrence is either vanishing for short distances or exponentially suppressed with r. The behavior of ξ_{2SE} confirms the peculiarity of the critical point of the isotropic model among those where the ground state of the system gets factorized. We understand such peculiarity as related with the fact that the critical point of the isotropic model does in fact coincide with saturation, i.e. with a special case of factorization. Therefore, while peculiar features of the entanglement properties and standard magnetic properties are usually observed for different values of the field, h_f and h_c respectively, in the isotropic model they occur together at the saturation field. A significant consequence of this feature is that, due to the lack of anisotropy in the XY plane, the XX model has $g_r^{xx} = g_r^{yy}$ and the correlation lengths along the x- and y-directions are consequently identical: therefore, the two-spin entanglement length, that we have found to equal the correlation length along the y-direction, in the isotropic, $\gamma = 0$, model coincides with the relevant correlation length along the x-direction, which consequently shares the divergence at $h_c = h_f$.

5. Residual entanglement and entanglement ratio

The amount of the entanglement stored between two spins far apart in the chain does not only rely on the distribution of concurrences $\{C_r\}$, but also on the total entanglement of the system and on whether it is bipartite or multipartite. A simple way to investigate this issue is to evaluate the one-tangle τ_1, the residual tangle $\tau_1 - \tau_2$ and the relative weight of the pairwise entanglement through the entanglement ratio τ_2/τ_1, in the neighborhood of the factorized circle. For $0 < \gamma \leqslant 1$, using the definitions of one- and two-tangle and the expressions for the magnetizations and concurrences, one obtains

$$\tau_1 = \frac{(1-\gamma)(3+\gamma)}{8\gamma^3(1+\gamma)}\varepsilon^2 + O(\varepsilon^3), \tag{26}$$

$$\tau_1 - \tau_2 = \frac{(1-\gamma)^2(2+\gamma)}{8\gamma^3(1+\gamma)}\varepsilon^2 + O(\varepsilon^3), \tag{27}$$

$$\frac{\tau_2}{\tau_1} = \frac{(1+\gamma)^2}{3+\gamma} + O(\varepsilon). \tag{28}$$

For $\gamma = 0$, as noticed above, the correlators do not decay exponentially with r, but they rather follow a power law, as a consequence of the quasi-long-range order characterizing the ground state for $h < 1$ (compare equations (9)–(11) with equations (18) and (19)). This behavior, which reflects on the concurrences, equation (21), makes it cumbersome to evaluate the sum in the two-tangle expression. In fact, it is not difficult to show that $\tau_2 \propto \varepsilon^{1/2}$, but in order to obtain the proportionality coefficient with good accuracy one should retain several terms in the small ε expansion of C_r. For this reason, we preferred to numerically compute τ_2 and τ_2/τ_1 close to the factorizing field up to $\varepsilon = 10^{-6}$ and we verified that the first term of equation (28) also holds for $\gamma = 0$.

From the analysis of the above expressions, we note that the larger the anisotropy the higher the relative weight of the pairwise entanglement close to the factorized ground state. In particular, in the pure Ising limit $\gamma = 1$ the entanglement in the ground state is totally stored in pairwise form up to order ε^2, but both the total entanglement and the residual tangle are strongly suppressed, being $\tau_1 = \varepsilon^4/32 + O(\varepsilon^6)$ and $\tau_1 - \tau_2 = \varepsilon^6/64$. In the opposite limit, the small values of the anisotropy γ favor the presence of multipartite entanglement and reduce the relative weight of the two-spin entanglement. In order to reconcile this last statement with the fact that, as shown in section 3.2, a smaller anisotropy implies larger C_r, one should note that, as the anisotropy of the model decreases, the one-tangle becomes larger and larger and, for $\gamma = 0$, it is

$$\begin{aligned}\tau_1 &= 1 - 4M_z^2 \\ &= \frac{4\sqrt{2}}{\pi}\varepsilon^{1/2} - \frac{8}{\pi^2}\varepsilon + O(\varepsilon^{3/2}),\end{aligned} \tag{29}$$

where M_z is given by equation (20). Thus, even if in the $\gamma \to 0$ limit the fraction of pairwise entanglement reduces, the two-spin entanglement takes advantage of the overall increase of the total amount of the entanglement stored in the ground state.

6. Conclusions

In this paper, we studied how the two-spin entanglement is distributed along the spin chain described by the Hamiltonian (1), focusing our attention on the possible connections between the pairwise entanglement spreading and the symmetry of the model. In particular, we have considered the neighborhood of factorized ground states, where all the concurrences get finite, no matter the distance between the two spins, and the range of the concurrence diverges (see equations (17) and (22)). We have derived closed analytical expressions for correlators and concurrences in the neighborhood of the factorized circle: using these formulae we have derived an analytical expressions for the range of the concurrence R, and shown that, whenever R diverges, a characteristic length scale ξ_{2SE}, equation (23), naturally emerges in the system. This two-spin entanglement length, which defines the pure exponential decay of C_r, evidenced by equation (13), is finite for finite anisotropy and diverges as the isotropic critical point is approached, where it in fact coincides with the magnetic correlation length on the plane perpendicular to the applied field.

Our description shows how the two-spin entanglement distribution along the chain evolves, while moving from the Ising ($\gamma = 1$) to the isotropic ($\gamma = 0$) model: in fact, for $\gamma \to 1$, it results that a finite amount of the entanglement can be stored between nearest neighbor spins only; in addition, in the pure Ising case the range of the concurrence is always finite. On the other hand, the fully isotropic exchange interaction of the $\gamma = 0$ model evidently also favors the entanglement rearrangement between distant spins.

Remarkably enough, despite the concurrence is either vanishing for short distances or exponentially suppressed with r in any point of the parameter plane $h - \gamma$, in the isotropic case, close to $h = h_c = 1$ all the concurrences $\{C_r\}$ are finite and their value is independent of the specific pair of spins along the chain whatever the distance r between them. This fact together with the change of the divergence character of the concurrence range testifies the special role played by the critical point of the isotropic model in the distribution of the entanglement between two spins of the system.

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